CSE 150A-250A AI: Probabilistic Models

Lecture 15

Fall 2025

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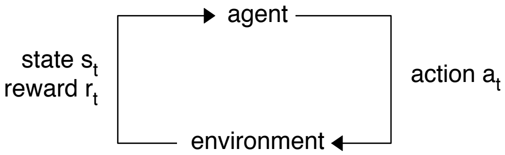
Slides adapted from previous versions of the course (Prof. Lawrence, Prof. Alvarado, Prof Berg-Kirkpatrick)

Agenda

Review

Reinforcement learning (RL)

- Learning from experience in the world



- Formalization as Markov decision process

$\mathcal{S}$	state space
$\mathcal{A}$	action space
$P(s'\|s,a)$	transition probabilities
$R(s)$	reward function
MDP	$\{\mathcal{S}, \mathcal{A}, P(s'\|s,a), R(s)\}$

- Definition

  A **policy** $\pi : \mathcal{S} \rightarrow \mathcal{A}$ is a mapping of states to actions.
  In this class we will only consider deterministic policies.

$\pi(s)$.

Decision-making in MDPs

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- **Number of policies**

  If there are $|\mathcal{A}|$ possible actions in each of $|\mathcal{S}|$ states, then there are *combinatorially* many policies:

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- **Experience under policy π**

$$\begin{array}{lccccccc} \text{state} & s_0 & \xrightarrow{\quad\pi(s_0)\quad} & s_1 & \xrightarrow{\quad\pi(s_1)\quad} & s_2 & \cdots \\ \text{reward} & r_0 & & r_1 & & r_2 & \cdots \end{array}$$

  Transitions occur with probabilities $P(s'|s, \pi(s))$.

Test your understanding

deterministic

A policy π completely determines the next state s' that an agent will end up in after taking an action from state s.

True (A) or False (B)?

How to measure long-term return?

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These are the most obvious ways to accumulate rewards.
But they are **not** the most commonly used in practice …

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What does it mean when the discount factor $\gamma << 1$?

A. Immediate and future rewards are valued equally.

B. Future rewards are heavily discounted compared to immediate.

C. Future rewards are lightly discounted compared to immediate.

D. Only future rewards are considered.

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   When $\gamma \ll 1$, future rewards are heavily discounted.
   These returns can be optimized by short-sighted agents.

   When γ is close to 1, future rewards are lightly discounted.
   These returns can only be optimized by far-sighted agents.

Motivation for $\gamma \in [0, 1)$

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   Discounted returns lead to simple iterative algorithms with strong guarantees of convergence.

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Maximizing the expected return is:

- generally wiser than maximizing the best-case return,
- but not as robust as minimizing the worst-case return.

Value functions

State value function

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- Values versus rewards:

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- **Values versus rewards:**

  The reward $R(s)$ give immediate feedback to the agent.

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  Sacrifice now for long-term gain: $R(s) < 0$, $V^{\pi}(s) > 0$.

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  Sacrifice now for long-term gain: $R(s) < 0$, $V^{\pi}(s) > 0$.
  Win now at the expense of later: $R(s) > 0$, $V^{\pi}(s) < 0$.

Properties of the state value function

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- Experience under policy π

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$$\text{state} \quad s_0 \xrightarrow{\pi(s_0)} s_1 \xrightarrow{\pi(s_1)} s_2 \cdots$$

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The **Bellman equation** tells us how.

Bellman equation

$$V^{\pi}(s) \quad =$$

Bellman equation

$$V^{\pi}(s) = \mathrm{E}^{\pi}\left[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots \,\middle|\, s_0 = s\right]$$

$$=$$

Bellman equation

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&=
\end{aligned}
$$

markov prop + Law of total expectation.

$s_0 = s$

$s_1 = s'$, $s_0 = s$

Value at s'

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The Bellman equation is the basis for much that will follow:

$$
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  Useful to imagine how small changes affect expected outcomes.

not ness true

$\pi(S) \to a$

$\to a \neq \pi(S)$

after $\downarrow$ $\pi(S)$

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  (All optimal policies share the same value functions.)

Relations at optimality

- From the optimal action value function:

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This is the problem of **planning** in MDPs.

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  Why *greedy*? Because we change the action in state s to whatever appears to improve the expected return.

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$$\begin{aligned} \pi'(s) &= \underset{a}{\text{argmax}} \left[Q^{\pi}(s,a) \right] \\ &= \underset{a}{\text{argmax}} \left[R(s) + \gamma \sum_{s'} P(s'|s,a) \, V^{\pi}(s') \right] \\ &= \underset{a}{\text{argmax}} \left[\sum_{s'} P(s'|s,a) \, V^{\pi}(s') \right] \end{aligned}$$

- Test your understanding:

$\pi'(s) = \pi(s)$ for some $s \in \mathcal{S}$? not necessarily

$\pi'(s) \neq \pi(s)$ for some $s \in \mathcal{S}$? not necessarily

$Q^{\pi}(s, \pi'(s)) \geq Q^{\pi}(s, \pi(s))$ for all $s \in \mathcal{S}$?

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Policy improvement

Policy improvement

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$$\pi'(s) = \operatorname*{argmax}_a Q^\pi(s, a)$$

Policy improvement

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$$\pi'(s) \quad = \quad \underset{a}{\operatorname{argmax}}\; Q^{\pi}(s, a)$$

- **Theorem:**
  The greedy policy $\pi'(s) = \arg\max_a Q^{\pi}(s, a)$ improves everywhere on the policy π from which it was derived:

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  If it's better to choose action a in state s before following π, then it's always better to make this choice.

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- **Intuition:**
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- **Proof idea:**
  We'll prove a key inequality for *one-step deviations* from π, then we'll extend this inequality by an iterative argument.

Proof — 1. Deriving the inequality

Proof — 1. Deriving the inequality

- Comparing value functions:

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- Intuition:

  It is better to take one step under π', then revert to π, than to always follow π.

Proof — 2. Leveraging the inequality

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Proof — 3. Taking the limit

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  It is better to follow π' (always) than to follow π (always).

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- **Take the limit $t \to \infty$:**

  It is better to follow π' (always) than to follow π (always). Conclude that $V^{\pi}(s) \leq V^{\pi'}(s)$ for all states $s \in \mathcal{S}$.

Policy iteration

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Policy iteration is guaranteed to terminate.

True (A) or False (B)?

Policy iteration

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- **Theorem**

  If $\pi'(s) = \arg\max_a Q^\pi(s,a)$ and $V^{\pi'}(s) = V^\pi(s)$ for all $s \in \mathcal{S}$,

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Policy iteration

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- **Proof idea**

Prove a key equality/inequality for terminal/non-terminal policies; iterate t times, then compare the limits as $t \to \infty$.

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- Suppose policy iteration converges to π'.

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These equations are nonlinear due to the max operation.

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Now exploit that π' is greedy with respect to π ...

- **Bellman optimality equation**

$$\boxed{V^{\pi}(s) = R(s) + \gamma \max_a \sum_{s'} P(s'|s,a) V^{\pi}(s')}$$

These equations are nonlinear due to the max operation.
There are n equations for n unknowns (where $s = 1, 2, \ldots, n$).

- Let $\tilde{\pi}$ be any policy of the MDP:

Proof — 2. Inequality

- Let $\tilde{\pi}$ be any policy of the MDP:

$$V^{\tilde{\pi}}(s) \quad = \quad R(s) + \gamma \sum_{s'} P(s'|s, \tilde{\pi}(s)) V^{\tilde{\pi}}(s') \qquad \boxed{\text{Bellman equation}}$$

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V^{\tilde{\pi}}(s) &= R(s) + \gamma \sum_{s'} P(s'|s, \tilde{\pi}(s)) V^{\tilde{\pi}}(s') && \boxed{\text{Bellman equation}} \\
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- Compare to Bellman optimality equation (BOE):

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  The **BOE** only holds for a solution π from policy iteration.

Proof — 3. Taking the limit

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Both right sides agree up to term of order γ^t.

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Since $\tilde{\pi}$ is arbitrary, we conclude that π is optimal.

That's all folks!